

Scenario-based Capital Requirements for the Interest Rate Risk of Insurance Companies

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28th April 2017

Abstract

The Solvency II standard formula measures interest rate risk based on two stress scenarios which are supposed to reflect the 1-in-200 year event over a 12-month time horizon. The calibration of these scenarios appears much too optimistic when comparing them against historical yield curve movements. This article demonstrates that interest rate risk is measured more accurately when using a (vector) autoregressive process together with a GARCH process for the residuals. In line with the concept of a pragmatic standard formula, the calculation of the Value-at-Risk can be boiled down to 4 scenarios, which are elicited with a Principal Component Analysis (PCA), at the cost of a relatively small measurement error.

Keywords: Interest Rate Risk, Principal Component Analysis, Capital Requirements, Solvency II

JEL classification: G17, G22, G32, G38

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1 Introduction

Interest rate risk is one of the most significant risks of insurance companies. Since insurers are facing long-term obligation, they invest over long time horizons, and a large portion of their assets are fixed income investments such as bonds or mortgage loans. Typically, the durations of assets and liabilities are not matched, but life insurers attain much longer durations on their liability side than on their asset side.¹ Modern insurance regulation frameworks, such as Solvency II in the European Economic Area, impose risk-based capital requirements to address those risks. Under Solvency II, the capital requirement is defined as the 99.5% Value-at-Risk of the change in economic capital over one year.² Most insurers determine the capital requirement with a standard formula. Regarding interest rate risks, the standard formula applies multiplicative stress factors to the current yield curve to determine an upward and a downward shift of interest rates, and insurers need to recalculate their capital in these scenarios. The calibration of the stress factors, at least for the downward scenario, appears much too optimistic. Between 1999 and 2015, the downward stress scenario underestimated the drop in interest rates during the subsequent 12 months for periods in 2011 as well as between 2014 and 2015 (EIOPA, 2016, p. 59). This poor backtesting result is broadly in line with the result of Gatzert and Martin (2012), who show that the standard formula's risk assessment of bond investments is inappropriate when comparing it against a partial internal model. Apart from the calibration, the standard formula may systematically underestimate the risk of curvature changes of the yield curve. Since both stress scenarios reflect yield curve shifts, insurers

¹EIOPA (2014, p. 17) report that durations of liabilities are on average 10 years longer than those of assets for Austrian, German, Lithuanian and Swedish insurers.

²Cf. European Commission (2009), Art. 101 (3).

can immunize against them by closing the duration gap.³ The capital requirement may thereby drop (close) to zero. However, insurers are still vulnerable to changes in the curvature.

The objective of this article is to derive a model for interest rate risk which is applicable for a regulatory standard formula. The model should fulfill the following properties:

1. The model should allow for determining the Solvency II capital requirement (i.e. 99.5% Value-at-Risk with a 1-year holding period).
2. The capital requirement should pass backtesting against historical yield curve movements.
3. The capital requirement should reflect the insurer's exposure to curvature changes of interest rates.
4. It should take into account that there is a lower bound to interest rates reflecting the economic costs of storing cash.
5. It should be a pragmatic approach, which insurers can implement for example by recalculating their economic equity capital in a small number of scenarios.

In the first part of this paper, we develop a model that is intended to fulfill properties 1-4. To this end, two models stand as candidates. The first candidate model focuses on the 4 parameters of the Nelson and Siegel (1987) model or 6 parameters of the extension by Svensson (1994) which both represent the whole yield curve. Diebold and Li (2006) derive good forecasts of future yield curves by modeling the development of these parameters

³Cf., e.g., Litterman and Scheinkman (1991, p. 55 f.) who explain how to construct a portfolio that is immunized against a particular movement (not necessarily parallel) of the yield curve.

over time with stochastic processes. In particular, the forecasts outperform those of affine factor models, which we therefore do not use in this paper.⁴ Caldeira et al. (2015) build on the procedure of Diebold and Li (2006) to estimate the Value-at-Risk of fixed income portfolios with a 1-day holding period. For longer holding periods, in particular when being situated in a low yield environment, it becomes relevant whether the model could simulate arbitrarily high negative interest rates or accounts for a lower bound (cf. property 4 above). Unfortunately, we are not aware of an easy extension of the approach from Caldeira et al. (2015) to incorporate a lower bound.⁵

As a second candidate model, we suggest a two-step approach which is particularly designed to respect such a lower bound. In a first step, the interest rates for 5 key maturities are modeled, each as the logarithmic distance to a lower bound. The development of the 5 key maturity interest rates over time is described by a (vector) autoregressive process together with a multivariate GARCH process for the residuals. In a second step, we use the Svensson model to interpolate and extrapolate the 5 interest rates to the complete term structure.

We backtest the Value-at-Risk estimates of both candidate models for relatively high confidence levels and relatively long holding periods against historical data. To address properties 2 and 3, the backtesting is conducted for 1,000 portfolios whose payoffs and maturities are randomly chosen.⁶ For a given portfolio, the accuracy of the Value-at-Risk is measured by the portion of historical time windows in which the Value-at-Risk under-

⁴The weak performance of affine factor models to forecast the yield curve has also been demonstrated by Duffee (2002). Moreover, by modeling the short rate, affine factor models focus on yield curve shifts and may therefore underestimate the risk of curvature changes when being used for capital requirements.

⁵Eder et al. 2014 propose incorporating a lower bound for interest rates by means of a plane-truncated normal distribution. However, the approach is numerically extensive and it therefore seems difficult to combine it with a GARCH process to address heteroscedasticity in longer time horizons.

⁶The backtesting in this case is more challenging than the one performed by Caldeira et al. (2015), who focus on the Value-at-Risk of equally-weighted portfolios.

estimates the loss in value that the portfolio had experienced for the actual change in interest rates (hit rate). Across all portfolios, the accuracy of the proposed 2-step model is similar to the dynamic Svensson model. In the context of capital requirements and resulting incentives for risk management, the model's accuracy should not substantially vary across portfolios (otherwise, regulation provides opportunities for regulatory arbitrage, since firms are incentivized to realize portfolios with risks that are measured too optimistically). In this regard, the proposed two-step model performs better than the dynamic Svensson model, since the standard deviation of the hit rate across portfolios is smaller. Moreover, from a regulatory perspective, it is important that the Value-at-Risk exceedances are not concentrated on particular time periods. Again, the proposed model performs slightly better than the Svensson model.

In the second part of the paper, we derive a scenario-based approximation of the Value-at-Risk in order to address property 5. The scenarios are elicited by a principal component analysis from the simulated yield curves according to the model in part 1. When the proposed two-step model is used for this purpose, the scenarios respect lower bounds for interest rates as well. The Value-at-Risk is determined by calculating the loss in capital for each scenario and then aggregating these losses by a square-root formula. A backtesting with 100,000 randomly chosen portfolios demonstrates that the 99.5% Value-at-Risk with a 1-year holding period can be closely approximated based on 4 yield curve stress scenarios.

The remainder of the paper is organized as follows. Section 2 outlines the approaches for stochastically modeling interest rate risks (part 1) and transforming the Value-at-Risk into a scenario-based calculation (part 2). Section 3 calibrates the models based on yield curve data published by the European Central Bank (ECB). Section 4 provides the

backtesting of the stochastic models (part 1) as well as of the scenario-based calculation (part 2). Section 5 concludes.

2 Value-at-Risk for interest rate risk

2.1 Firm model

We consider an insurance company that expects future cash inflows $A_1, \dots, A_M \geq 0$ from assets and outflows $L_1, \dots, L_M \geq 0$ from insurance obligations in $1, 2, \dots, M$ years, where M denotes the largest maturity under consideration. The expected surpluses, $S_\tau = A_\tau - L_\tau$ for $\tau = 1, \dots, M$, are collected in a column vector $S = (S_1, \dots, S_M)^T$. The firm's equity value in time 0 (i.e. the interest-rate-sensitive part of it) is obtained as the present value of the surpluses:

$$E_0 = \sum_{\tau=1}^M e^{-\tau \cdot r_0(\tau)} S_\tau, \quad (1)$$

where $r_0(\tau)$ is the continuously compounded risk-free interest rate for maturity τ at time 0. Let $r_h(\tau)$ denote the stochastic interest rate for maturity τ in h years and the random variable

$$E_{0,h} = \sum_{\tau=1}^M e^{-\tau \cdot r_h(\tau)} S_\tau, \quad (2)$$

the firm's equity value if interest rates instantaneously shift to $r_h(\tau)$. The Value-at-Risk for interest rate risk for confidence level $1 - \alpha$ and holding period h is obtained as

$$\text{VaR}_{1-\alpha,h} = -q_\alpha(E_{0,h} - E_0), \quad (3)$$

where $q_\alpha(X)$ denotes the α -quantile of the random variable X . The focus of this paper is the Solvency II requirement for interest rate risk, which is determined as $\text{VaR}_{99.5\%}^{IR}(1 \text{ year})$.

2.2 Modeling interest rate risk

We consider two models for interest rates. On the one hand, we use a dynamic version of the model from Svensson (1994). According to this model, the continuously compounded interest rate at time t for maturity τ is represented by

$$\begin{aligned} r_t(\tau) = & c_{1,t} + c_{2,t} \cdot \frac{1 - e^{-\tau/\lambda_{1,t}}}{\tau/\lambda_{1,t}} + c_{3,t} \cdot \left[\frac{1 - e^{-\tau/\lambda_{1,t}}}{\tau/\lambda_{1,t}} - e^{-\tau/\lambda_{1,t}} \right] \\ & + c_{4,t} \cdot \left[\frac{1 - e^{-\tau/\lambda_{2,t}}}{\tau/\lambda_{2,t}} - e^{-\tau/\lambda_{2,t}} \right] \end{aligned} \quad (4)$$

where $\theta_t = (c_{1,t}, c_{2,t}, c_{3,t}, c_{4,t}, \lambda_{1,t}, \lambda_{2,t})$ is a vector with time-varying parameters.⁷ The model is used by central banks, such as the European Central Bank (ECB), to elicit a yield curve out of bond market data. The ECB estimates and publishes the parameters on a daily basis and determines the yield curve accordingly.

On the other hand, we model interest rates with a simple exponential model. For each maturity in a predefined set, $\tau \in (\tau_1, \dots, \tau_m)$, the interest rate is modeled as

$$r_t(\tau) = e^{\theta_t(\tau)} + r^{\min}(\tau) \quad (5)$$

where $\theta_t = (\theta_t(\tau_1), \dots, \theta_t(\tau_m))$ is considered as a vector of time-varying parameters, and $r^{\min}(\tau)$ is a lower bound for the interest rate for maturity τ .

⁷In contrast to Caldeira et al. (2015), we consider $\lambda_{1,t}$ and $\lambda_{2,t}$ also as time-varying parameters. This allows us to avoid a residual term in Eq. 4 given that the model will be calibrated based on daily data for the 6 parameters of the Svensson model in section 3.

For both models (Eq. 4 and 5), the development of the parameter vectors θ_t over time may exhibit autocorrelation. We address autocorrelation using two alternative approaches. Firstly, we assume that the development of each entry depends only on the history of that entry. Hence, each entry $\theta_t^{(i)}$ follows an autoregressive (AR) stochastic process:

$$\Delta\theta_t^{(i)} = \mu_i + \sum_{k=1}^{p_i} \gamma_{k,i} \cdot \theta_{t-k}^{(i)} + \eta_{t,i}, \quad (6)$$

where $\Delta\theta_t^{(i)} = \theta_t^{(i)} - \theta_{t-1}^{(i)}$, $p_i \in \mathbb{N}$ is the lag order of the process for entry i , $\mu_i, \gamma_{k,i} \in \mathbb{R}$ are constant coefficients, and the stochastic process $\eta_{t,i}$ reflects the disturbances. To describe the movement of all parameters, we need 6 processes in case of the Svensson model and m processes in case of the exponential model.

Secondly, we consider a vector-autoregressive (VAR) model, in which the development of each entry $\theta_t^{(i)}$ depends on the history of all entries:

$$\Delta\theta_t = \mu + \sum_{k=1}^p \Gamma_k \cdot \theta_{t-k} + \eta_t, \quad (7)$$

where $\Delta\theta_t = \theta_t - \theta_{t-1}$, $p \in \mathbb{N}$ is the lag order of the VAR process, $\mu \in \mathbb{R}^m$ (with $m = 6$ in case of the Svensson model), the Γ_k are $m \times m$ transition matrices, and the multivariate stochastic process η_t reflects the disturbances. Note that Equation 7 can also be used to denote the AR model by collecting the $\gamma_{k,i}$ in diagonal matrices Γ_k .

Finally, the disturbances process η_t may exhibit time-varying volatilities and correlations, which we address by means of the dynamic conditional correlation (DCC) model proposed

by Engle (2002). In this model, the covariance matrix Ω_t is decomposed into a time-varying correlation matrix R_t and a $m \times m$ diagonal matrix D_t such that

$$\Omega_t = D_t R_t D_t. \quad (8)$$

Using

$$z_t = D_t^{-1} \eta_t, \quad (9)$$

the η_t are transformed into $(m \times 1)$ -vectors z_t of uncorrelated, standardized disturbances with mean zero and variance one. The elements in the correlation matrices R_t are denoted by $\rho_{i,j,t}$ and obtained as

$$\rho_{i,j,t} = \frac{q_{i,j,t}}{\sqrt{q_{i,i,t}q_{j,j,t}}}, \quad (10)$$

where the $q_{i,j,t}$ are the elements of $m \times m$ matrices Q_t . The diagonal matrix D_t and the matrix Q_t follow GARCH-like processes:

$$D_t^2 = \text{diag}(\omega_i) + \text{diag}(\kappa_i) \circ \eta_{t-1} \eta_{t-1}^T + \text{diag}(\lambda_i) \circ D_{t-1}^2 \quad (11)$$

$$Q_t = (1 - a - b) \bar{Q} + a z_{t-1} z_{t-1}^T + b Q_{t-1} \quad (12)$$

where $\text{diag}(x_i)$ generates a $m \times m$ diagonal matrix with x_1, \dots, x_m on the diagonal, \circ denotes the Hadamard product, \bar{Q} is the unconditional covariance matrix, $\omega_i, \kappa_i, \lambda_i$ are non-negative parameters $\forall i \in \{1, \dots, m\}$, and a, b are non-negative parameters such that $a + b < 1$.

For the Svensson model, each realization of the modeled parameter vector θ_t can be translated directly into the complete yield curve by applying Eq. 4. In case of the exponential model, modeling the interest rates for maturities τ_1, \dots, τ_m is only the first step. In the

second step, the parameters of the Svensson model are fitted to each realization of the vector of interest rates.⁸ Then Eq. 4 is applied to receive a realization of the complete yield curve.

In total, we consider 4 models by combining the Svensson model (“Sve”) and the two-step exponential model (“Exp”) with the AR process and the VAR process.

2.3 Transformation into a scenario-based approach

The result of the procedure in section 2.2 is a calibrated multivariate stochastic process for the parameter vector θ_t . This can be used to generate a large number of simulations for the values of the parameters over a time horizon of length h (e.g. one year). After applying Eq. 4 to elicit the complete yield curve in every simulation path, the Value-at-Risk is determined according to Eq. 14.

For a standard formula, however, this procedure might not be an appropriate method, since complex information (i.e. the modeled yield curve in a large number of simulations) would need to be provided by the regulator. The aim of this section is to approximate the Value-at-Risk with a simplified calculation method, in order to reduce the information that the regulator needs to provide to a small number of scenarios.

The starting point for this purpose is the fact that the portfolio losses are linear in the discount factors, and therefore the discount factors are the actual risk drivers. Consider an insurance company with expected surpluses $S = (S_{\tau_1}, \dots, S_{\tau_K})^T$ at maturities τ_1, \dots, τ_K

⁸For simplicity, we take the values for $\lambda_{1,t}$ and $\lambda_{2,t}$ from the last observed point in time and only determine $c_{1,t}, \dots, c_{4,t}$, which can be fitted by OLS.

and let X_1 denote the random vector with the discount factors corresponding to interest rates for those maturities in 1 year:

$$X_1 = \left(e^{-r_{1 \text{ year}}(\tau_1)}, \dots, e^{-r_{1 \text{ year}}(\tau_K)} \right)^T, \quad (13)$$

Moreover, let $\mathbb{E}(X_1)$ denote its expectation and X_0 the corresponding deterministic vector of discount factors at time 0. Then, the Value-at-Risk for interest rate risk is obtained as

$$\text{VaR}_{1-\alpha, 1 \text{ year}} = -\left(q_\alpha(X_1^T \cdot S) - X_0^T \cdot S \right), \quad (14)$$

In order to reduce the required information for this calculation, we transform X_1 into its principal components such that

$$X_1 = \Theta \cdot Y + \mathbb{E}[X_1]. \quad (15)$$

By construction of the PCA, Y is a random vector of order K with $\mathbb{E}[Y] = 0$, the covariance matrix of which is a diagonal matrix. We can recalculate the Value-at-Risk in line 14 as

$$\begin{aligned} & -\left(q_\alpha((\Theta \cdot Y + \mathbb{E}[X_1])^T \cdot S) - X_0^T \cdot S \right) \\ &= -q_\alpha((\Theta \cdot Y)^T \cdot S) - (\mathbb{E}[X_1] - X_0)^T \cdot S \\ &= -q_\alpha\left(\sum_{k=1}^K Y_k \cdot (\Theta^T \cdot S)_k \right) - (\mathbb{E}[X_1] - X_0)^T \cdot S \end{aligned} \quad (16)$$

Here, $(\Theta^T \cdot S)_k$ is the k -th entry of the vector $(\Theta^T \cdot S)$ and reflects the insurer's exposure to the k -th principal component. Assume for a moment that X_1 (and hence Y) follows a

multivariate normal (or Student's t) distribution, the standardized marginal distributions of which have the α -percentile z_α . Then, the Value-at-Risk in line 16 can be determined by

$$\sqrt{\text{Var}\left(\sum_{k=1}^K Y_k \cdot (\Theta^T \cdot S)_k\right)} \cdot z_\alpha - (\mathbb{E}[X_1] - X_0)^T \cdot S \quad (17)$$

Since the covariance matrix of Y is diagonal, this can be rewritten as

$$\sqrt{\sum_{k=1}^K \text{Var}(Y_k) \cdot (\Theta^T \cdot S)_k^2} \cdot z_\alpha - (\mathbb{E}[X_1] - X_0)^T \cdot S \quad (18)$$

Under the given distribution assumption, we have

$$\text{Var}(Y_k) \cdot (\Theta^T \cdot S)_k^2 \cdot z_\alpha^2 = \left(\underbrace{q_\alpha(Y_k \cdot (\Theta^T \cdot S)_k)}_{\text{VaR}_k}\right)^2 \quad (19)$$

The quantile on the right-hand side of Eq. 19 measures the risk related to principal component k , which we denote by VaR_k . Irrespectively of the distribution assumption of Y , we can rewrite VaR_k by pulling out the factor $(\Theta^T \cdot S)_k$:

$$\begin{aligned} \text{VaR}_k &= \begin{cases} q_\alpha(Y_k) \cdot (\Theta^T \cdot S)_k & \text{if } (\Theta^T \cdot S)_k \geq 0 \\ q_{1-\alpha}(Y_k) \cdot (\Theta^T \cdot S)_k & \text{if } (\Theta^T \cdot S)_k < 0 \end{cases} \\ &= \begin{cases} \sum_{\tau=1}^K (X_{0,\tau} + q_\alpha(Y_k) \cdot \Theta_{\tau,k}) \cdot S_\tau - X_0 \cdot S & \text{if } (\Theta^T \cdot S)_k \geq 0 \\ \sum_{\tau=1}^K (X_{0,\tau} + q_{1-\alpha}(Y_k) \cdot \Theta_{\tau,k}) \cdot S_\tau - X_0 \cdot S & \text{if } (\Theta^T \cdot S)_k < 0 \end{cases} \end{aligned} \quad (20)$$

where $X_{0,\tau} = e^{-r_0(\tau)}$ denotes the τ -th element of X_0 . The expressions $X_{0,\tau} + q_\alpha(Y_k) \cdot \Theta_{\tau,k}$ and $X_{0,\tau} + q_{1-\alpha}(Y_k) \cdot \Theta_{\tau,k}$ in line 20 can be comprehended as discount factors for maturity τ years and related to principal component k . Based on

$$r^{k,1}(\tau) = -\ln[X_{0,\tau} + q_\alpha(Y_k) \cdot \Theta_{\tau,k}] / \tau \quad (21)$$

and

$$r^{k,2}(\tau) = -\ln[X_{0,\tau} + q_{1-\alpha}(Y_k) \cdot \Theta_{\tau,k}] / \tau, \quad (22)$$

$\tau = \tau_1, \dots, \tau_K$, they can be translated into “stressed” interest rates related to that principal component. Hence, the Value-at-Risk related to principal component k is calculated as the change in own funds when the yield curve changes in a stress scenario.

In order to receive the total Value-at-Risk, the results for VaR_k are aggregated as in Eq. 18. Since the first few components can already explain a large share of the variation, a good approximation of the Value-at-Risk might already be achieved by only taking the first $\tilde{K} < K$ components into account. Section 4.2 demonstrates that \tilde{K} can be reduced to 2 without a major deterioration in accuracy. Moreover, we cut negative results, which could occur due to the approximation:

$$\text{VaR}_{1-\alpha}(1 \text{ year}) \approx \max \left\{ \sqrt{\sum_{k=1}^{\tilde{K}} \text{VaR}_k^2} - (\mathbb{E}[X_1] - X_0)^T \cdot S; 0 \right\} \quad (23)$$

3 Calibration of interest rate models

3.1 Data

The model calibration is based on data published by the ECB, which estimates the yield curve from AAA-rated euro area central government bonds with the Svensson method.⁹ The ECB publishes the 6 parameters of the Svensson model, together with the corresponding interest rates for maturities 1, 5, 10, 20 and 30 years, for every trading day since 6 September 2004. We use these data on a daily basis from 6 September 2004 to 30 December 2016, giving us 3156 observations of the yield curve. Table 1 shows the descriptive statistics for the daily changes in interest rates for maturities 1, 5, 10, 20 and 30 years.

Table 1: Descriptive statistics for daily changes in interest rates from 6.9.2004 to 30.12.2016 (in basis points).

Maturity	Mean	Std Dev.	Min	Max	Skewness	Kurtosis
1	-0.099	2.600	-26.368	19.440	-0.714	14.530
5	-0.125	3.899	-22.579	18.312	0.029	5.126
10	-0.125	3.911	-19.305	18.054	0.164	4.574
20	-0.124	4.276	-24.100	24.027	0.032	5.854
30	-0.124	4.942	-56.403	31.075	-0.307	11.306

3.2 Calibration

For the exponential model, we choose to model $m = 5$ maturities 1, 5, 10, 20 and 30 years in line with the interest rates published by the ECB. For all maturities, the lowest possible interest rates $r^{min}(\tau)$ are set to -3% . The largest maturity under consideration is set to $M = 40$ years.

⁹Cf. <https://www.ecb.europa.eu/stats/money/yc/html/index.en.html>

In the observed time horizon from 2004 to 2016, interest rates have significantly decreased (cf. column “Mean” in Table 1). We remove this drift from the observed $\Delta\theta_t$ -processes (for the dynamic Svensson model as well as for the exponential model) by deducting the mean of $\Delta\theta_t^{(i)}$ for each entry i . This helps to avoid the negative drift continuing in the simulated yield curves of the next year, which would drive interest rates in 1 year below the current level in expectation.

The lag orders of the AR and VAR-processes are chosen according to the criterion of Hannan and Quinn (1979) (HQ).¹⁰ Table 2 shows that the optimal lag orders p^* of the 4 models under consideration range between 1 and 3.

Table 2: Optimal lag orders p^* for each entry $\theta_t^{(i)}$ (AR-process) or the vector θ_t (VAR-process) according to HQ criterion.

	AR						VAR
Svensson	c_1	c_2	c_3	c_4	λ_1	λ_2	
p^*	3	1	1	1	1	1	1
Exp	$\tau = 1$ year	5 years	10 years	20 years	30 years		
p^*	2	1	2	2	2		2

To estimate the parameters μ and Γ of the AR and VAR-model, OLS regression per equation is used. The parameters of the DCC model are estimated with R software using the rmgarch package from Ghalanos (2015).¹¹ For the exponential model, the entries of the disturbances vectors z_t in Eq. 9 are assumed to be standard normally distributed. Since the estimation of the Svensson model parameters does not lead to stable results when the standard normal distribution is used for the entries of z_t , we choose for each of the 6 entries of z_t between the standard normal, Student’s t and the generalized error

¹⁰Shittu and Asemota (2009) demonstrate that this criterion outperforms the Akaike information criterion (AIC) and the Bayesian information criterion (BIC) for autoregressive processes and large samples.

¹¹The estimation works in two steps. In the first step, the parameters $\omega_i, \kappa_i, \lambda_i$ of Eq. 11 are determined by Maximum-Likelihood estimation. Using the predictions for D_t according to Eq. 11, z_t is determined based on Eq. 9. In the second step, a and b of Eq. 12 are estimated.

distribution. To this end, the HQ criterion is calculated for each of the $3^6 = 729$ possible combinations of the three distributions. Table 3 depicts the chosen marginal distributions which maximize the HQ criterion.

Table 3: Marginal distributions of z_t -process in the Svensson model maximizing the HQ criterion; choice between standard normal (norm), Student's t (std) and generalized error distribution (ged).

Entry k of z_t	1	2	3	4	5	6
AR-model	ged	std	ged	norm	norm	norm
VAR-model	ged	std	std	ged	norm	std

4 Backtesting

4.1 Backtesting interest rate models

We backtest the Value-at-Risk calculation according to the 4 models by comparing them with losses that would have been realized for the historical yield curve movements. Since the realized loss depends on the composition of the portfolio (which may exhibit long or short exposures for each maturity), the analysis is carried out for 1,000 randomly generated asset-liability portfolios of hypothetical insurance companies. Each portfolio $i \in \{1, \dots, 1000\}$ consists of 5 payoffs. The maturity of each payoff was chosen by a discrete uniform distribution on the set $\{1, 2, 3, \dots, 40\}$. Each expected cash flow $CF_\tau^{(i)}$ was chosen by a uniform distribution on the interval $[-1, 1]$. All these choices were conducted independently from each other.

The descriptive overview of the portfolios in Table 4 shows that for about 50% of the model insurers, the present value of liabilities exceeds the present value of (interest-rate-sensitive) assets. This does not mean that these companies are already insolvent, since

they may own further assets which are not valued based on a present-value-calculus. Moreover, about 50% of the model insurers exhibit a negative duration gap in terms of the absolute duration, i.e. the duration of liabilities is longer than that of assets, as it typically applies for life insurers. The other 50% of portfolios exhibit a positive duration gap, meaning that the duration of assets is longer than that of liabilities, as typically happens for non-life insurers. The Macaulay duration informs that the duration gap of the median insurer amounts to about 18 years. Given the quite considerable duration gaps, the set of backtesting portfolios should challenge the models for interest rate risk at least as much as the actual portfolios of, for example, European insurance companies in the market.

Table 4: Descriptive overview of 1,000 randomly chosen portfolios for the backtesting.

Percentile	Present value	Absolute duration	Macaulay duration
25%	-0.754	-15.425	6.113
50%	-0.015	0.063	18.086
75%	0.776	14.113	27.667

The Solvency II capital requirement is the 99.5% Value-at-Risk with a 1-year holding period. Backtesting this value with historical data is impossible, since it would require interest rate data from at least 200 years. Instead, we conduct the backtesting for several combinations of holding periods h (in trading days) and confidence levels $1 - \alpha$ and check whether an increase in the holding period h systematically affects the accuracy of the Value-at-Risk estimate.

The backtesting is carried out as an in-sample fit, i.e. μ and Γ are estimated once based on the whole sample.¹² Using each of the 4 models calibrated above, we determine the

¹²This set-up is chosen to simplify the analysis and since its focus lies on the model's appropriateness to measure the risk consistently across portfolios and stably over time, rather than on the estimation of the parameters.

Value-at-Risk $\text{VaR}_{(1-\alpha),h}^i(t)$ at day t and for portfolio i . To this end, we generate 5,000 simulations of $\eta_t, \eta_{t+1}, \dots, \eta_{t+h-1}$ in accordance with the covariance matrices $\Omega_t, \dots, \Omega_{t+h-1}$. Then, the parameters $\theta_{t+1}, \dots, \theta_{t+h}$ (Eq. 7), the interest rates $r_{t+1}(\tau), \dots, r_{t+h}(\tau)$ (Eq. 4 and 5) and the Value-at-Risk of each portfolio i (Eq. 14) are determined. The Value-at-Risk $\text{VaR}_{(1-\alpha),h}^i(t)$ is compared with the historical loss that has occurred for that portfolio between times t and $t+h$:

$$\text{loss}_t^{(i)} = - \sum_{\tau=1}^M (e^{-\tau \cdot r_{t+h}(\tau)} - e^{-\tau \cdot r_t(\tau)}) \cdot CF_{\tau}^{(i)} \quad (24)$$

In order to avoid autocorrelation in the $\text{loss}_t^{(i)}$ -processes, we conduct this comparison only beginning at every h -th day of the observed time period. We can thereby observe $n = \lfloor \frac{3156-3}{h} \rfloor$ pairwise disjunct time windows, each with a length of h days.¹³ The percentage of days for which the historical loss exceeds the Value-at-Risk is called the hit rate:

$$\text{hit rate} = \frac{1}{n} \sum_t \mathbb{1}_{\{\text{loss}_t^{(i)} > \text{VaR}_{(1-\alpha),h}^i(t)\}} \quad (25)$$

For an accurate Value-at-Risk estimate for portfolio i , the hit rate should be close to α , i.e. for about $\alpha \cdot n$ days, the historical loss should exceed the Value-at-Risk.

We conduct the analysis for $\alpha = 0.5\%$, $\alpha = 5\%$ and $\alpha = 10\%$ combined with holding periods of $h = 1$, $h = 15$, $h = 30$ and $h = 50$ days. Note that sampling error affects the hit rate more strongly the smaller $\alpha \cdot n$ is. Hence, the hit rate of the 99.5% Value-at-Risk is estimated relatively robustly for $h = 1$, but it is quite vulnerable to sampling error

¹³The number of observed realizations of the yield curves is reduced by 3, since we focus on yield curve changes, and the first 2 observations are needed to kick off the process.

when $h = 50$. For long holding periods, only the hit rate of the 90% Value-at-Risk is relatively robust.

Table 5: Averages and standard deviations of hit rates across 1,000 random portfolios.

Model	Holding period (in days)	$\alpha = 1 -$ confidence level					
		0.5%		5%		10%	
		Average	Std Dev.	Average	Std Dev.	Average	Std Dev.
Sve - AR	1	0.60%	0.43%	3.37%	1.52%	6.16%	2.36%
	15	2.16%	2.48%	8.94%	5.98%	13.02%	7.35%
	30	2.13%	3.18%	10.57%	7.08%	15.38%	8.60%
	50	1.69%	3.52%	12.19%	8.23%	17.81%	9.16%
Sve - VAR	1	0.45%	0.32%	2.62%	0.98%	4.78%	1.38%
	15	1.08%	0.88%	4.47%	2.32%	7.58%	2.97%
	30	0.25%	0.72%	3.38%	3.02%	7.19%	4.21%
	50	0.10%	0.60%	1.77%	2.51%	4.77%	3.48%
Exp - AR	1	0.56%	0.39%	3.95%	0.50%	8.20%	0.64%
	15	0.75%	0.59%	4.18%	1.19%	8.79%	1.48%
	30	0.92%	1.07%	4.51%	2.22%	8.11%	2.63%
	50	0.24%	0.60%	4.32%	1.69%	8.44%	3.34%
Exp - VAR	1	0.51%	0.38%	3.76%	0.52%	8.08%	0.61%
	15	0.87%	0.67%	4.74%	1.71%	9.37%	2.09%
	30	1.33%	1.45%	5.10%	2.27%	9.38%	3.00%
	50	0.37%	0.81%	5.85%	2.61%	10.72%	4.10%

Table 5 shows the averages and standard deviations of the hit rates across the 1,000 randomly chosen portfolios for each of the 4 models. For a 1-day holding period ($h = 1$) and the highest confidence level ($\alpha = 0.5\%$), all four models are relatively accurate for the portfolios on average. The Sve-VAR model appears to estimate the Value-at-Risk slightly too conservatively (average hit rate = 0.45%), while the Sve-AR model appears slightly too optimistic (average hit rate = 0.60%).

To check whether the models allow for a consistent risk aggregation over time, we focus on the 90% Value-at-Risk, for which the hit rate is also robust for longer holding periods. The corresponding hit rates for the Sve-AR model continuously increase from 6.16% for $h = 1$ to 17.81% for $h = 50$. Hence, the Value-at-Risk estimate of this model appears to become more and more optimistic the longer the holding period is. A similar (although less severe) pattern is observed for the hit rates of the Exp-VAR model, whose hit rates

continuously increase from 8.08% to 10.72%. In contrast, the hit rates of the Sve-VAR model and of the Exp-AR model do not systematically increase or decrease for longer time periods, in line with a consistent risk aggregation over time.

As explained in the introduction, it is important from a regulatory perspective that the model's accuracy is stable across portfolios, which is measured by the variation of hit rates across the portfolios. Since the standard deviation of the hit rate is systematically higher the higher the average hit rate is, we focus on the coefficients of variation. From the two models which consistently aggregate risk over time (Sve-VAR and Exp-AR), the Exp-AR model exhibits the lower coefficients of variation. For instance, for the 99.5%-Value-at-Risk, the coefficients of variation of the Exp-AR model amount to $0.39\%/0.56\% = 69\%$ for $h = 1$, 79% for $h = 15$, 116% for $h = 30$, and 248% for $h = 50$. The corresponding values of the Sve-VAR model amount to 72%, 81%, 294%, and 634%. The Exp-AR model also exhibits lower coefficients of variation when looking at the hit rates of the 95%-Value-at-Risk or of the 90%-Value-at-Risk for each of the holding periods considered.

Finally, we demonstrate that the Exp-VAR model performs better than the Sve-VAR model in terms of the independence of Value-at-Risk exceedances over time. For each portfolio and each combination of h and α , we employ the independence test of Christoffersen (1998) on the null hypothesis that a Value-at-Risk exceedance does not affect the probability of an exceedance in the subsequent time window. Table 6 shows the portion of portfolios for which the pattern of Value-at-Risk exceedances is significantly dependent over time with p-values below 1%, 5% and 10%. Regarding the Exp-AR model for the 99.5%-Value-at-Risk, significant time dependencies occur only for the shortest holding period of $h = 1$ day. In this case, for 3.2% of portfolios the exceedances are time-dependent at a 1% level of significance. For the Sve-VAR model, the corresponding

portion of portfolios is much higher (16.2%). Here, significant dependencies also persist for Value-at-Risks with longer holding periods. Looking at the 90%-Value-at-Risk, the test can make use of a larger sample, since more exceedances are expected. In this case, the Sve-VAR creates much more significant time dependencies of exceedances than the Exp-AR model, particularly for longer holding periods.

Table 6: Portion of portfolios with p-value of Christoffersen’s Exceedance Independence test below 1%, 5%, and 10%.

	$\alpha = 1 - \text{Confidence level of Value-at-Risk}$								
	0.5%			5%			10%		
p-value below	1%	5%	10%	1%	5%	10%	1%	5%	10%
<u>Exp-AR model:</u>									
<u>Holding period</u>									
1 day	3.2%	12.9%	22.4%	2.7%	10.2%	15.5%	5.2%	11.8%	16.1%
15 days	0.0%	0.0%	0.0%	2.1%	13.9%	16.6%	1.0%	9.7%	23.5%
30 days	0.0%	0.0%	0.0%	0.0%	0.3%	0.9%	0.0%	0.7%	3.2%
50 days	0.0%	0.0%	0.0%	0.0%	0.1%	0.1%	0.0%	0.0%	0.2%
<u>Sve-VAR model:</u>									
<u>Holding period</u>									
1 day	16.2%	49.5%	59.8%	70.7%	88.6%	92.1%	92.0%	97.9%	98.9%
15 days	4.7%	6.5%	8.7%	42.8%	47.9%	53.9%	46.1%	50.5%	53.6%
30 days	6.1%	9.3%	11.1%	40.9%	50.8%	56.3%	64.5%	73.7%	79.4%
50 days	1.4%	2.1%	3.4%	58.8%	68.2%	71.8%	67.0%	75.2%	80.2%

In total, the backtesting reveals that the Exp-AR model gives the best Value-at-Risk estimates in terms of the accuracy of the 99.5% Value-at-Risk, the risk aggregation over time, the variation of the accuracy across portfolios and the independence of exceedances. Consequently, this model is used going forward to determine the 99.5% Value-at-Risk for 1 year and to derive the scenario-based calculation.

4.2 Backtesting the scenario-based Value-at-Risk

The starting point and benchmark for the scenario-based approach are simulations of the yield curves after 1 year. We calibrate the Exp-AR model as on 30 December 2016

and generate 30,000 simulations for the interest rates $r_{\text{year end 2017}}(\tau, \omega)$ for maturities $\tau \in \{1, 5, 10, 20, 30\}$.¹⁴ For each simulation path, we fit the parameters of the Svensson model to the five modeled interest rates.¹⁵ Then Eq. 4 is applied to obtain 30,000 simulations of the whole yield curve.

To set up the scenarios, we transform the simulated interest rates $r_{\text{year end 2017}}(\tau, \omega)$ for the 5 maturities $\tau \in \{1, 5, 10, 20, 30\}$ into principal components (hence, $K = 5$ in Eq. 15). We then use Eq. 21 and 22 to elicit two stressed yield curves for each principal component. Each of these scenarios is translated into a complete yield curve by fitting the Svensson parameters to the 5 stressed interest rates.¹⁶ Finally, the Value-at-Risk is calculated according to Eq. 23.

Table 7 depicts the stressed interest rates for the maturities 1, 5, 10, 20 and 30 years in terms of absolute changes to the interest rates on 30 December 2016. Regarding the first principal component (PC1), the two stress scenarios *A* and *B* are essentially an upward and downward shift of interest rates. The second principal component (PC2) affects the steepness of the yield curve by changing long-term interest rates in a different direction than short and middle-term rates. At the same time, it impacts the bowing of the yield curve, since middle-term interest rates are changed more strongly than short and long-term yields. In terms of the third principal component, only scenario *B* has an essential effect on the yield curve. This scenario impacts the curvature, since interest rates for the maturities 10 and 30 years change in a different direction than those for the others.

¹⁴We assume that 1 year has 270 trading days.

¹⁵As mentioned in section 2.2, we take the values for $\lambda_{1,t}$ and $\lambda_{2,t}$ from 30.12.2016 and only fit $c_{1,t}, \dots, c_{4,t}$ by OLS.

¹⁶Again, $\lambda_{1,t}$ and $\lambda_{2,t}$ are taken from 30.12.2016 and $c_{1,t}, \dots, c_{4,t}$ are fit by OLS.

Table 7: Interest rate stress scenarios (in absolute changes to yield curve on 30 December 2016), PCA applied to modeled interest rates for maturities 1, 5, 10, 20 and 30 years.

		Maturity				
		1 year	5 years	10 years	20 years	30 years
PC1	A	-0.3%	-0.9%	-1.4%	-1.9%	-1.9%
	B	0.3%	0.9%	1.6%	3.1%	4.7%
PC2	A	-0.3%	-0.6%	-0.8%	-0.1%	0.2%
	B	0.4%	0.9%	1.2%	0.1%	-0.2%
PC3	A	0.0%	-0.1%	0.0%	0.0%	0.0%
	B	0.3%	0.3%	-0.2%	0.2%	-0.1%
PC4	A	-0.6%	-0.6%	0.1%	0.1%	0.0%
	B	0.4%	0.5%	0.0%	-0.1%	0.0%
PC5	A	-0.9%	0.0%	0.0%	0.0%	0.0%
	B	1.4%	0.0%	0.0%	0.0%	0.0%

We backtest the scenario-based Value-at-Risk by comparing it against the exactly calculated Value-at-Risk, which is based on the whole simulation. To this end, we generate 100,000 random portfolios by the same process that has been used for the portfolio generation in section 4.1. Figure 1 depicts the exact Value-at-Risk (x axis) and the scenario-based Value-at-Risk (y axis) for the 100,000 portfolios. The y coordinates of the black points have been calculated based on the four scenarios of PC1 and PC2. The close fit between the x and y coordinates illustrates that the scenario-based method provides a good approximation for all portfolios. The y coordinates of the gray points were calculated with the two scenarios of PC1 only. In this case, the scenario-based method underestimates the risk for some portfolios, which, however, mostly contain relatively little interest rate risk. If scenarios of further principal components (PC3, PC4 or PC5) were taken into account, the scatter plot would look like the black points in Figure 1, meaning that these scenarios do not substantially enhance the accuracy of the approximation.

Figure 1: Value-at-Risk of 100,000 simulated portfolios according to exact method (x-axis) and scenario-based calculation (y-axis).

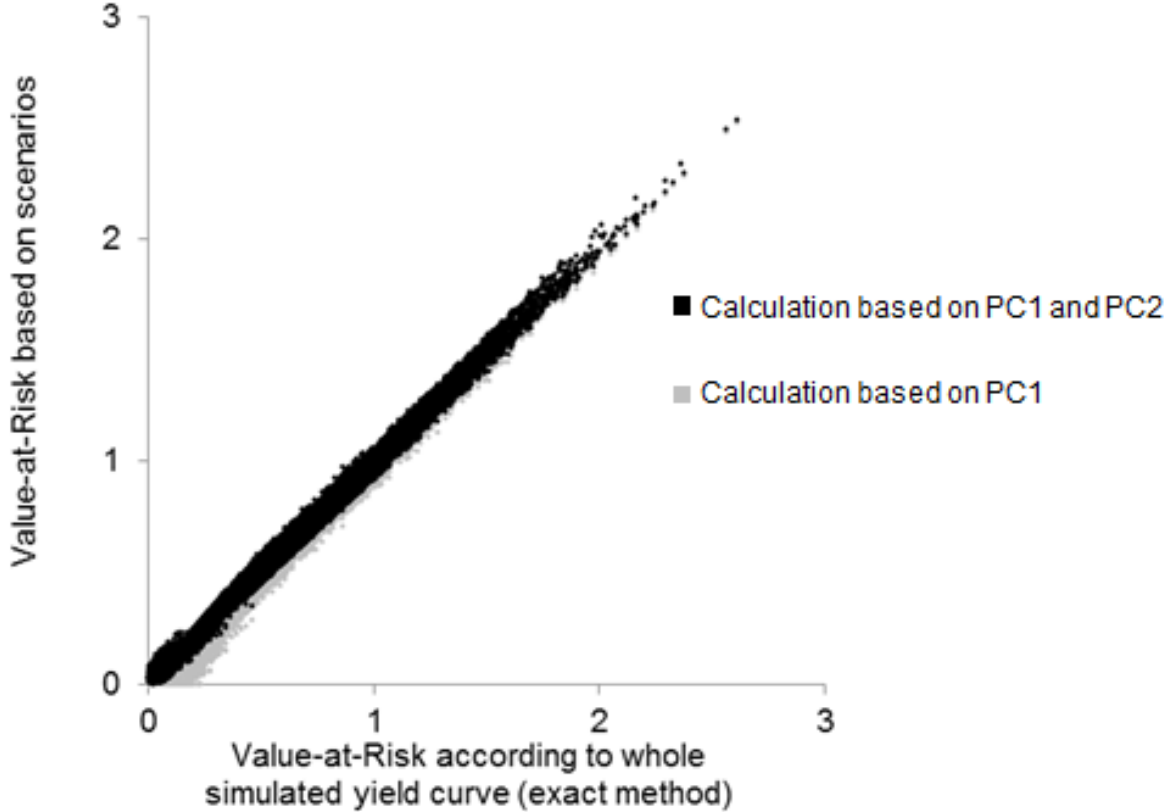


Table 8 shows the mean absolute error (MAE) of the scenario-based calculation in comparison to the exact Value-at-Risk across the 100,000 portfolios (the first row of Table 8 relates to Figure 1 and takes all data between 2004 and 2016 into account). The error is reduced by about 29% from 0.0220 to 0.0155 when taking the scenarios of PC2 in addition to those of PC1 into account. Further scenarios do not enhance the accuracy much; including more scenarios than those of PC1 - PC3 even slightly increases the MAE.

To verify the robustness of the sound approximation based on PC1 and PC2, we recalculate for all 100,000 portfolios the exact and the scenarios-based Value-at-Risk based on data from 2004 until the end of year x , with $x \in \{2012, \dots, 2015\}$. The Value-at-Risk refers to the end of year x , with a holding period of $h = 1$ year and $\alpha = 0.5\%$. The results are shown in rows 2 to 5 of Table 8. Using the scenarios of PC2 in addition to those of PC1 reduces the MAE between 38% and 47% in the additional four calibrations. For the

calibration until 2013, the scenarios of PC3 further reduce the MAE by 8%; for all other calibrations the effect does not exceed 1%.

Table 8: Mean absolute error (MAE) of scenario-based Value-at-Risk in comparison to exact Value-at-Risk for 100,000 portfolios. Numbers in brackets show the relative change in MAE by including an additional principal component.

Data	Number of principal components, \bar{K}									
	1		2		3		4		5	
2004-2016	0.0220	0.0155	(-29%)	0.0155	(-1%)	0.0156	(+1%)	0.0156	(±0%)	
2004-2015	0.0232	0.0142	(-39%)	0.0140	(-1%)	0.0141	(±0%)	0.0141	(±0%)	
2004-2014	0.0276	0.0170	(-38%)	0.0169	(±0%)	0.0169	(±0%)	0.0170	(±0%)	
2004-2013	0.0204	0.0107	(-47%)	0.0098	(-8%)	0.0098	(±0%)	0.0098	(±0%)	
2004-2012	0.0254	0.0135	(-47%)	0.0133	(-1%)	0.0133	(±0%)	0.0134	(±0%)	

Finally, a historical backtesting supports the validity of the scenario-based Value-at-Risk. Analogously to section 4.1, we compare the scenario-based Value-at-Risk for 1,000 randomly chosen portfolios with historical losses. For each h -day time window between 6.9.2004 and 30.12.2016, we simulate interest rates with the Exp-AR model, transform the simulated interest rates by a PCA and elicit yield curve scenarios (cf. Eq. 21 and 22). The averages and standard deviations of the hit rates for the scenario-based Value-at-Risk shown in Table 9 are close to the hit rates of the accurate calculation of the Exp-AR model shown in Table 5.

Table 9: Averages and standard deviations of hit rates for scenario-based Value-at-Risk across 1,000 random portfolios.

Holding period	$\alpha = 1 - \text{Confidence level}$					
	0.5%		5%		10%	
	Average	Std Dev.	Average	Std Dev.	Average	Std Dev.
1 day	0.56%	0.40%	3.79%	0.54%	8.07%	0.76%
15 days	0.79%	0.66%	4.16%	1.25%	8.76%	1.51%
30 days	0.92%	1.08%	4.50%	2.27%	8.08%	2.75%
50 days	0.22%	0.57%	4.34%	1.70%	8.46%	3.41%

5 Conclusion

When determining a method for the definition and monitoring of capital requirements for financial institutions, regulators traditionally need to deal with a trade-off: on the one hand, they could force the entities to develop and use an internal model, which might measure risk appropriately by building on recent market data modeling risks with Monte-Carlo simulations. However, this requires complex processes on the firms' side to implement and maintain the models as well as on the regulator's side to supervise the models. On the other hand, capital requirements can be formulated by a pragmatic standard formula, which, however, typically comes along with mismeasurements which can create severe disincentives for risk management decisions. This paper suggests a procedure to translate a Monte-Carlo-based risk measurement of interest rate risk into a small number of scenarios. The scenarios offer a way to define capital requirements by a pragmatic calculation rule, the result of which is close to the Monte-Carlo-based calculation. The suggested procedure utilizes PCA, which is applied to the simulations of a stochastic model, rather than directly to historical data. By doing so, additional requirements, such as a lower bound for interest rates, can be included. Apart from the applicability in the context of a regulatory standard formula, the scenarios can build a basis for regulatory or enterprise-internal stress tests.¹⁷ Moreover, the idea of translating Monte-Carlo simulations into scenarios could be generally helpful for interpreting and communicating the results of an interest rate risk model, which is often challenging due to the large number of simulation paths.

¹⁷This is also required in bank risk management, cf. Bank for International Settlements (2016, p. 7-10).

Further research is needed on the question of how to deal with interest rates for very long maturities, which cannot be stably estimated from bond market data. Moreover, follow-up research topics include the integration of further market risks into the scenario-based approach.

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