Mixed Insurance as Optimal Policy under Rejoicing Sensitivity*

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Abstract:
This study investigates the mixed insurance, which contains both death benefit (coverage) and living benefit (bonus), under the rejoicing theory. In this study, we analyze the situation in which individuals endogenously determined the coverage and bonus in mixed insurance by incorporating rejoicing sensitivity into Raviv (1979).

The main results of this study are as follows. First, under the expected utility theory, mixed insurance is never an optimal policy under the expected utility theory. Second, under rejoicing-sensitive preference, mixed insurance is an optimal policy. Third, a raise in rejoicing sensitivity leads to decrease coverage and increase bonus. Last, the threshold probability, which realizes same amounts of coverage and bonus, is decreasing function of rejoicing sensitivity.

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1. Introduction

In many studies in insurance economics, life insurance firms sell life insurance products and individuals purchase those that pay death benefit (coverage) when the insured is dead before the end of the term of insurance. In other words, “term (death) insurance” is (implicitly) assumed in many insurance economic models. Then, individual can receive the coverage only when the insured is dead, while he cannot receive any benefit when the insured is still living at the end of the term of insurance.

The policyholders are worried about decrease in their income when the insured is dead. For preparing that decrease in income, individuals want to purchase term insurance that provides coverage for decreasing in their income when the insured is dead. In other words, the main purpose of purchasing term insurance is to alleviate the income uncertainty. Actually, individuals can perfectly dissolve that income uncertainty through purchasing full insurance.

In contrast, some life insurance products have not only coverage but also living benefit (bonus). This kind of life insurance products are called as “mixed insurance” that pay either coverage or bonus before or end of the term of insurance. Endowment insurance is one of the well-known life insurance products categorized in mixed insurance. Actually, endowment insurance is widely distributed in Japanese life insurance market that is the second largest life insurance market in the world. According to Life Insurance Fact Book 2016 published by The Life Insurance Association of Japan (URL: http://www.seiho.or.jp/english/statistics/trend/pdf/2016.pdf), the number of new policies endowment insurance is occupied in about 9.2 percent in FY 2015. Thus, we believe that mixed insurance cannot be negligible in insurance economics.

However, mixed insurance has almost been ignored in literatures in insurance economics. One of the main reasons that did not focus on mixed insurance is that mixed insurance cannot be justified under expected utility theory which is a dominant tool in insurance economics. For simplicity, suppose the individuals who demand the coverage for decrease in their income in the case of death. Also suppose that insurance firm sells both term and mixed insurance when both insurance premiums are actuarially fair. In this situation, individuals surely purchase full insurance in term insurance because they want to perfectly dissolve their income uncertainty. In contrast, mixed insurance is never chosen because some degree of income uncertainty remains.

From that viewpoint, we prospect that other methodology must be needed for justifying the presence of mixed insurance in actual life insurance market. One of the advantages in mixed insurance is that policyholders surely receive either coverage or bonus from insurance firm. This advantage has two sides of implication.

First side of the implication is that the policyholder's regret, which might be felt at the end of the term of insurance, toward purchasing life insurance leads to lower. In the case of term insurance, policyholders have great regret when the insured was not died because they cannot receive any benefit. In contrast, in the case of mixed insurance, policyholders have small (no) regret because they can receive bonus when the insured was not died. Thus, incorporating the regret into expected utility theory, which is called "regret theory", is one of the promising solutions for justifying the presence of mixed insurance.

The pioneering studies in regret theory are Bell (1982) and Loomes and Sugden (1982). Braun and Muermann (2004) is the first study to analyze insurance market under regret theory. Recent studies that analyze insurance market under regret are Huang et al. (2014, 2015).

However, these previous studies were not investigated mixed insurance. The first and only study to analyze mixed insurance under regret theory is Fujii et al. (2016). Fujii et al. (2016) shed light on how the regret affects the individuals' choice in insurance product and derive the condition in which individuals chose mixed insurance rather than term insurance.

Second side of the implication is that the policyholder's rejoicing, which might be felt at the end of the term of insurance, toward purchasing life insurance leads to rise. In the case of term insurance, policyholders cannot have rejoicing when the insured was not died because they cannot receive any benefit. In contrast, in the case of mixed insurance, policyholders have rejoicing because they can receive bonus when the insured was not died. Thus, incorporating the rejoicing into expected utility theory, which is called "rejoicing theory", is another promising solution for justifying the presence of mixed insurance.

Then, we have to know which theory is more appropriate for analyzing mixed insurance. About this question, Fujii et al. (2016) insists that rejoicing theory is more appropriate in terms of death probability. In the case of rejoicing theory, the necessary condition of presence of mixed insurance is that death probability is less than 1/2. In contrast, in the case of regret theory, the necessary condition of presence of mixed insurance is that death probability is more than 1/2. Thus, we find that rejoicing theory is more appropriate theory for investigating mixed insurance. Also, we have a brief evidence. According to the life table in 2015 (URL: http://www.mhlw.go.jp/english/database/db-hw/lifetb22nd/dl/tables.pdf), death probability is surely less than 1/2 unless the age of insured is over 100. In the end, this study is along with rejoicing theory.

The purpose of this study is to investigate the mixed insurance under rejoicing theory. This study incorporates rejoicing sensitivity into Raviv (1979), while Fujii et al. (2016) was based on Mossin (1968). The differences in foundational studies are directly related to the differences in coverage and bonus in mixed insurance. This study analyzes the situation in which individuals endogenously determine the coverage and bonus in mixed insurance, while these are exogenous variables in Fujii et al. (2016).

The organization of this article is as follows. In section 2, we provide a basic set-up of the model under expected utility theory. In section 3, rejoicing-sensitive preference is represented. In section 4, an optimal policy is derived under rejoicing-sensitive preference. In sections 5 and 6, we investigate the properties of optimal insurance and derive the main propositions of this study. Concluding remarks are in Section 7.

2. Optimal Policy under Expected Utility Theory

An insured endows an identical initial wealth w and faces a risky loss. For simplicity, the risky loss is assumed to describe a binary probability distribution. The loss, $D > 0$, is occurred with probability $p \in (0,1)$ and is not occurred with probability $1 - p$. A risk-neutral insurer offers an insurance policy from which the insured receives $I(D)$ in the loss state and $I(0)$ in the no-loss state. The payment in the loss state, $I(D)$, is called a coverage and that in the no-loss state, $I(0)$, is a bonus. It is assumed that provision of the payments do not incur any costs. The following inequality is assumed to be satisfied for the coverage and the bonus:

$$0 \leq I(0) \leq A,$$
$$0 \leq I(D) \leq D.$$

The assumption, $0 \leq I(0) \leq A$, is essentially different from existing literature on optimal insurance policy. Previous studies suppose that the payment is not generated in no-loss state. This means $I(0) = 0$, that is, the possibility is excluded that mixed insurance becomes optimal policy.

Insurance premium ,Q, is assumed to be actuarially fair and it represents

$$Q = pI(D) + (1 - p)I(0).$$

Given the coverage and the bonus, the final wealth is written:

$$W_L = w - D - Q + I(D) \text{ and } W_{NL} = w - Q + I(0)$$

for the loss state and the no-loss state, respectively. Then, the final wealth is rewritten:

$$W_L = w - D + (1 - p)\Delta I \text{ and } W_{NL} = w - p\Delta I,$$

where $\Delta I = I(D) - I(0)$. This equation means that the difference between the coverage and the bonus is relevant to the level of the final wealth.

Under the expected utility theory, a risk averse insured, whose utility function is $u' > 0$ and $u'' < 0$, determines an optimal policy $\{I(D), I(0)\}$ to maximize the following optimization problem:

$$\max_{\{I(0),I(D)\}} pu\big(w - D - Q + I(D)\big) + (1-p)u\big(w - Q + I(0)\big)$$
$$s.t.\ \ Q = pI(D) - (1-p)I(0)$$
$$0 \leq I(0) \leq A \text{ and } 0 \leq I(D) \leq D.$$
.

The optimization problem can be rewritten:

$$\max_{\{I(0),I(D)\}} pu(w - D + (1-p)\Delta I) + (1-p)u(w - p\Delta I)$$
$$s.t.\ \ -A \leq \Delta I \leq D.$$

The first-order condition (FOC) is given

$$p(1-p)u'(W_L) - p(1-p)u'(W_{NL}) = 0$$
$$\Leftrightarrow u'(W_L) = u'(W_{NL}) \Leftrightarrow W_L = W_{NL}$$

The second-order condition is satisfied by $u'' < 0$. From $W_L = W_{NL}$, the optimal policy is hold at $\Delta I = D$ and finally we derive $\{I(D), I(0)\} = \{D, 0\}$. This result can be summarized in the following proposition:

Proposition 1.
Suppose a risk averse insured and a risk-neutral insurer. $\{I(D), I(0)\} = \{D, 0\}$ is an optimal insurance policy.

3. Rejoicing-sensitive Preference Representation

The regret-rejoicing sensitive preference is introduced by Bell (1982) and Loomes and Sugden (1982). However, their representation causes intransitivity when more than two alternatives are included. This means that it is difficult to find an optimal solution, and so their representation cannot directly be applied to most analyses. Braun and Muermann (2004) proposed a modified version where individual sets the highest wealth given a specific state as the foregone wealth which is compared with actual wealth. Thus, it can be applicable to a wide range of analysis because this modified form does not cause intransitivity. In this paper, another modified version is used for analysis, that is, individuals sets the lowest wealth as the reference wealth comparing with their actual wealth.

The utility function which represents the rejoicing-sensitive preference has the following form:

$$u(y) + kg\big\{u(y) - u\big(y^{\min}\big)\big\}.$$

This is called the rejoicing-sensitive utility function. Here, y denotes actual wealth and $y^{\min}$ denotes the highest wealth with respect to the specific state, which is either the loss-state or the no-loss state. For avoiding confusion of the terminology in this study, u was defined in the previous section is called risk function in the later discussion. $k \geq 0$ is called rejoicing coefficient. The rejoicing coefficient represents how insured puts a weight on rejoicing feeling. An insured whose coefficient is $k' > k$, is more rejoicing-sensitive compared an insured with k. When $k = 0$, the rejoicing-sensitive preference is degenerated into expected utility. We normalize $k = 1$ when the analysis can be applied to any positive rejoicing coefficients. g is called rejoicing function with $g(0) = 0$, $g' > 0$ and $g'' < 0$. The rejoicing-sensitive utility function consists of the two components. The first component is utility from actual wealth and the second component is utility from rejoicing by comparing the lowest wealth with actual wealth. The expected utility form of the rejoicing-sensitive preference is written:

$$E[u(\tilde{y}) + kg\{u(\tilde{y}) - u(\tilde{y}^{max})\}]$$

where the variables attached tilde represents random variables. Final wealth becomes a random variable because the expectation is defined by subjective probabilities over states. Also, the lowest wealth is random variable because it depends on the actual state.

4. Derivation of Optimal Policy under Rejoicing Sensitivity

The optimal policy is formulated by the two-stage optimization problem when the insured has rejoicing-sensitive utility function. First, ex-post decision problem, which includes the optimal policies given the specific states, are derived to determine the highest wealth. Second, the ex-ante decision problem is examined and the optimal policy is obtained.

4.1 The ex-post decision problem

For examining the optimal coverage with respect to the loss state, the objective function can be written as follows:

$$\max_{\Delta I} V(\Delta I) = u(w - D + (1-p)\Delta I)$$
$$s.t. -A \leq \Delta I \leq D \quad (1)$$

Ignoring the constraint, the FOC is given:

$$V'(\Delta I) = \frac{dV}{d\Delta I} = (1-p)u'(W_L) > 0 \quad (2)$$

From (2), the expected utility $V(\Delta I)$ is strictly increasing in ΔI. Thus, the expected utility is minimized when ΔI set the minimized value within the constraint (1). Then, we find that $\Delta I = -A$ achieves the lowest wealth. Because $0 \leq I(0) \leq A$ and $0 \leq I(D) \leq D$ is assumed, the optimal policy with respect to the loss state becomes

$\{I(D), I(0)\} = \{0, -A\}$. Substituting $\Delta I = -A$ to the objective function, the ex-post worst utility with respect to the loss state can be shown as

$$u(w - D + (1-p)\Delta I) = u(w - D - (1-p)A).$$

Then, the lowest wealth with respect to the no-loss state is examined. The objective function can be written as follows:

$$\max_{\Delta I} V(\Delta I) = u(w - p\Delta I)$$
$$s.t. \; -A \leq \Delta I \leq D$$

Applying a similar argument in the loss state, the FOC is given:

$$V'(\Delta I) = -u'(W_L) < 0. \quad (3)$$

From (3), the expected utility is decreasing in ΔI. Thus, the expected utility is minimized when ΔI set the maximum value with the constraint. Then, we find that $\Delta I = D$ achieves the lowest wealth. Because $-A \leq \Delta I \leq D$ is assumed, the optimal policy with respect to the no-loss state is given $\{I(D), I(0)\} = \{D, 0\}$. Substituting $\Delta I = D$ to the objective function, the ex-post worst utility with respect to the no-loss state can be shown as

$$u(w - p\Delta I) = u(w - pD).$$

4.2 The ex-ante decision problem

By using the lowest wealth in loss and no-loss states that were derived in previous section, the optimal policy can be determined by solving the following optimization problem:

$$\max_{\{\Delta I\}} V(\Delta I)$$
$$= p[u(w - D + (1-p)\Delta I) + g\{u(w - D + (1-p)\Delta I) - u(w - D - (1-p)A)\}]$$
$$+(1-p)[\{u(w) + g\{u(w - p\Delta I) - u(w - pD)\}\}$$
$$s.t. -A \leq \Delta I \leq D.$$

Ignoring the constraint, the FOC is given as:

$$V'(\Delta I) = p(1-p)u'(W_L)\big(1 + g'(UD_L)\big) - p(1-p)u'(W_{NL})\big(1 + g'(UD_{NL})\big) = 0.$$

Here

$$UD_L = u(w - D + (1-p)\Delta I) - u(w - D - (1-p)A) = u(w_L) - u\big(w_L^{\min}\big),$$
$$UD_{NL} = u(w - p\Delta I) - u(w - pD) = u(w_{NL}) - u\big(w_{NL}^{\min}\big).$$

The second-order condition is also satisfied by $u'' < 0$ and $g'' > 0$. The FOC can be rewritten:

$$u'(W_L)\big(1 + g'(UD_L)\big) = u'(W_{NL})\big(1 + g'(UD_{NL})\big)$$
$$\Leftrightarrow \frac{u'(W_L)}{u'(W_{NL})} = \frac{1 + g'(UD_{NL})}{1 + g'(UD_L)} \quad (4).$$

The optimal policy is satisfied the equality (4) with the constraint, $-A \leq \Delta I \leq D$.

5. Properties of Optimal Insurance

In this section, two properties of optimal insurance are stated in the form of two propositions.

Proposition 2.
Suppose a rejoicing-sensitive insured and a risk-neutral insurer. The optimal policy $\{I(D), I(0)\}$ is satisfied $-A < \mathit{\Delta} I < D$.

Proof:
The proof is divided into the following two parts, $\mathit{\Delta} I < D$ and $0 < \mathit{\Delta} I$.

In the first part, it is sufficient to show the sign of $\mathrm{V}'(\mathit{\Delta} I)$ at $\mathit{\Delta} I = D$, that is, $\{I(0), I(D)\} = \{0, D\}$. The final wealth becomes $W_L = W_{NL}$, then, $u'(W_L) = u'(W_{NL})$. Since $UD_{NL} = 0 < UD_L$, $g'(UD_{NL}) > g'(UD_L)$ is always realized because $g'' < 0$. Furthermore, from the equality (4), the following inequality is obtained.

$$\frac{u'(W_L)}{u'(W_{NL})} = 1 < \frac{1 + g'(UD_{NL})}{1 + g'(UD_L)}$$

$$\Leftrightarrow u'(W_L)\big(1 + g'(UD_L)\big) - u'(W_{NL})\big(1 + g'(UD_{NL})\big) < 0$$

This inequality indicates that $V'(\mathit{\Delta} I = D) < 0$. Thus, the optimal policy should be satisfied with $\mathit{\Delta} I < D$.

In the second part, it is sufficient to show the sign of $V'(\mathit{\Delta} I)$ at $\mathit{\Delta} I = -A$, , that is, $\{I(0), I(D)\} = \{A, 0\}$. Since $\mathit{\Delta} I = -A$ is lowest wealth with respect to the no-loss state, $UD_{NL} = 0$ holds. From $u'(W_L) > u'(W_{NL})$ and $g'(UD_L) > g'(UD_{NL})$ under $\mathit{\Delta} I = -A$, we have

$$V'(\Delta I = -A) = u'(W_L)\big(1 + g'(UD_L)\big) - u'(W_{NL})\big(1 + g'(UD_{NL})\big) > 0.$$

This inequality indicates that $V'(\Delta I = -A) > 0$. Thus, the optimal policy should be satisfied with $\mathit{\Delta} I > -A$.

Q. E. D.

Proposition 2 states that the optimal insurance holds in strict inequality. Under the rejoicing-sensitive utility function, the insured only feels rejoicing in the loss state since the full coverage is the lowest wealth with respect to the no-loss state. The insurance policy is marginally changed as the bonus $I(0)$ is increasing from zero and the coverage $I(D)$ is decreasing from D. From such marginal change, the insured feels more rejoicing in the loss state, but she feels less rejoicing in the no-loss state. Since the rejoicing function g is concave, the utility from increasing rejoicing in the no-loss state dominates the disutility from decreasing rejoicing in the loss state. This explanation implies that $\Delta I < D$ is an optimal policy. A similar argument can be applied in the case of $\Delta I > -A$.

Proposition 3: ΔI is decreasing in k.

Proof:
ΔI_k represents ΔI when the insured whose rejocing coefficient is k has an optimal policy. From the equality (4), the FOCs can be rewritten

$$\frac{u'(W_L)}{u'(W_{NL})} = \frac{1 + kg'(UD_{NL})}{1 + kg'(UD_L)}$$

Since $\Delta I < D$ for all $k > 0$, $W_L < W_{NL}$ and $u'(W_L) > u'(W_{NL})$ hold. From (4), $u'(W_L) > u'(W_{NL})$ coincides with $g'(UD_L) < g'(UD_{NL})$. Let us consider $k' > k$. Given at $\{I_k(0), I_k(x)\}$. In this situation, $g'(UD_L) < g'(UD_{NL})$ implies

$$\frac{u'(W_L)}{u'(W_{NL})} = \frac{1 + kg'(UD_{NL})}{1 + kg'(UD_L)} < \frac{1 + k'g'(UD_{NL})}{1 + k'g'(UD_L)}.$$

This inequality means

$$V'(\Delta I_k) = u'(W_L)\big(1 + k'g'(UD_L)\big) - u'(W_{NL})\big(1 + k'g'(UD_{NL})\big) < 0.$$

for an insured whose rejoicing coefficient is $k' > k$. As a result, $\Delta I_k > \Delta I_{k'}$ for $k' > k$ is confirmed.

Q. E. D.

Proposition 3 concludes that a raise in rejoicing sensitivity reduces the difference between the coverage and the bonus. Thus, we find that the insured whose rejoicing sensitivity is very high chooses similar amounts of coverage and bonus. From that viewpoint, endowment insurance, which includes same amount of coverage and bonus, might be an optimal policy for the insured whose rejoicing sensitivity is very high.

6. Loss Probability and Optimal Policy

In this section, we consider how loss probabilities influence the optimal policy. Let us define the threshold probability $\hat{p}$ such that $\Delta I = 0$ is the optimal policy. At $\Delta I = 0$,

$$\begin{aligned} &\operatorname{sgn}\left\{\frac{\partial}{\partial p}\left(\frac{1 + g'(UD_{NL})}{1 + g'(UD_L)}\right)\right\} \\ &= \operatorname{sgn}\left\{\begin{matrix} Du'\big(w_{NL}^{\min}\big)g''(UD_{NL})\big(1 + g'(UD_L)\big) \\ +Au''\big(w_{NL}^{\min}\big)g''(UD_{NL})\big(1 + g'(UD_L)\big)\end{matrix}\right\} < 0. \end{aligned}$$

This means that ΔI satisfies the single crossing property at $\hat{p}$ from the above. This is easily confirmed the following: if ΔI crosses twice, it is necessary to cross from the below. This contradicts the above inequality.

From the above argument, we obtain that

$$\frac{u'(W_L)}{u'(W_{NL})} > (<) \frac{1 + g'(UD_{NL})}{1 + g'(UD_L)}$$
$$\Leftrightarrow u'(W_L)1 + g'(UD_L) - u'(W_{NL})1 + g'(UD_{NL}) > (<)0$$
$$\Leftrightarrow \Delta I > (<)0.$$

for $\hat{p} < (>)p$. We summarize the argument into the following proposition:

Proposition 4:
Suppose that the optimal policy is $\Delta I = 0$ at the threshshold probability $\hat{p}$. The indemnity is more (less) than the bonus for higher (lower) loss probabilities than the threshold probability, $\Delta I > (<)0$ for $\hat{p} < (>)p$.

Last, we examine how the intensity of rejoicing sensitivity influences the threshold probabilities. To do so, we denote them $\hat{p}(k)$. Following a similar argument of proposition 3, we obtain the following inequality. Given $\hat{p}(k)$,

$$\frac{u'(W_L)}{u'(W_{NL})} = \frac{1 + kg'(UD_{NL})}{1 + kg'(UD_L)} < \frac{1 + k'g'(UD_{NL})}{1 + k'g'(UD_L)}$$

for $k < k'$. This means that the threshold probability is decreasing in k.

Proposition 5:
The threshold probability is decreasing in k, that is, $\hat{p}(k)$ is decreasing in k.

7. Concluding Remarks

This study investigated the mixed insurance under rejoicing theory. In this study, we analyzed the situation in which individuals endogenously determined the coverage and bonus in mixed insurance by incorporating rejoicing sensitivity into Raviv (1979).

The main results of this study are as follows. First, under the expected utility theory, an optimal policy only contains coverage. Thus, mixed insurance is never an optimal policy under the expected utility theory. Second, under rejoicing-sensitive preference, an optimal policy contains both coverage and bonus. Thus, mixed insurance is an optimal policy. Third, a raise in rejoicing sensitivity leads to decrease coverage and increase bonus. Thus, insured whose rejoicing sensitivity is very high chooses high amount of bonus. Last, the threshold probability, which realizes same amounts of coverage and bonus, is decreasing function of rejoicing sensitivity.

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